



EE 232 Lightwave Devices

Lecture 6: Electron-Photon Interaction, Optical Matrix Element, Absorption Coefficient

Instructor: Ming C. Wu

University of California, Berkeley
Electrical Engineering and Computer Sciences Dept.

EE232 Lecture 6-1

Prof. Ming Wu



Vector Potential

Maxwell's Equations

$$\left\{ \begin{array}{l} \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{D} = \rho \\ \nabla \cdot \vec{B} = 0 \end{array} \right.$$

$$\vec{D} = \epsilon \vec{E} \quad \text{and} \quad \vec{B} = \mu \vec{H}$$

Vector potential \vec{A}

$$\vec{B} = \nabla \times \vec{A}$$

$$\Rightarrow \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$$

Vector potential is not unique:

$$\vec{A}' = \vec{A} + \nabla \xi \rightarrow \nabla \times \vec{A}' = \vec{B}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{A} = \nabla \times \left(-\frac{\partial \vec{A}}{\partial t} \right)$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \phi$$

Wave Equations:

$$\left\{ \begin{array}{l} \left(\nabla^2 \vec{A} - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \vec{A} + \mu \epsilon \frac{\partial \phi}{\partial t} \right) = -\mu \vec{J} \\ \nabla^2 \phi + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon} \end{array} \right.$$

Wave Equations are invariant under

"Gauge Transformation"

$$\left\{ \begin{array}{l} \vec{A}' = \vec{A} + \nabla \xi \\ \phi' = \phi - \frac{\partial \xi}{\partial t} \end{array} \right. \Rightarrow \text{One can choose the value of } \nabla \cdot \vec{A}$$

EE232 Lecture 6-2

Prof. Ming Wu



Lorenz and Coulomb Gauges

Lorenz Gauge:

$$\text{Set } \nabla \cdot \vec{A} + \mu\epsilon \frac{\partial \phi}{\partial t} = 0$$

$$\begin{cases} \nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \\ \nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon} \end{cases}$$

Coulomb Gauge

$$\text{Set } \nabla \cdot \vec{A} = 0$$

$$\begin{cases} \left(\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \nabla \left(\mu\epsilon \frac{\partial \phi}{\partial t} \right) = -\mu \vec{J} \\ \nabla^2 \phi = -\frac{\rho}{\epsilon} \end{cases}$$

For optical field, $\rho = 0, \phi = 0$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}$$



Electron-Photon Interaction

Hamiltonian in the absence of light

$$H_0 = \frac{P^2}{2m_0} + V(\vec{r})$$

Generalized momentum of a charged particle in an electromagnetic field

$$\vec{P} \rightarrow (\vec{P} - e\vec{A})$$

Note: $\frac{\partial \vec{P}}{\partial t} = \text{Force} \rightarrow \left(\frac{\partial \vec{P}}{\partial t} - e \frac{\partial \vec{A}}{\partial t} \right) = \left(\frac{\partial \vec{P}}{\partial t} - e\vec{E} \right)$

(modification represents EM fields' ability to accelerate or decelerate electrons)

$$H = \frac{1}{2m_0} (\vec{P} - e\vec{A})^2 + V(\vec{r})$$

$$= \frac{P^2}{2m_0} - \frac{e}{2m_0} (\vec{P} \cdot \vec{A} + \vec{A} \cdot \vec{P}) + \frac{e^2 A^2}{2m_0} + V(\vec{r})$$

$$H' = -\frac{e}{2m_0} (\vec{P} \cdot \vec{A} + \vec{A} \cdot \vec{P}) + \frac{e^2 A^2}{2m_0}$$

$$H' \approx -\frac{e}{2m_0} (\vec{P} \cdot \vec{A} + \vec{A} \cdot \vec{P})$$

$$(\vec{P} \cdot \vec{A})\phi = -i\hbar \nabla \cdot \vec{A}\phi$$

$$= -i\hbar \left[(\nabla \cdot \vec{A})\phi + \vec{A} \cdot \nabla \phi \right]$$

$$= -i\hbar \vec{A} \cdot \nabla \phi \quad \text{under Coulomb Gauge}$$

$$= (\vec{A} \cdot \vec{P})\phi$$

$$H' = -\frac{e}{m_0} \vec{A} \cdot \vec{P}$$

• For a brief review of Hamiltonian classical mechanics, see http://en.wikipedia.org/wiki/Hamiltonian_mechanics#Using_Hamilton.27s_equations



Optical Matrix Element

$$H' = -\frac{e}{m_0} \vec{A} \cdot \vec{P}$$

$$\vec{A} = \hat{e} A_0 \cos(\vec{k}_{op} \cdot \vec{r} - \omega t) = \hat{e} \frac{A_0}{2} e^{i\vec{k}_{op} \cdot \vec{r} - i\omega t} + \hat{e} \frac{A_0}{2} e^{-i\vec{k}_{op} \cdot \vec{r} + i\omega t} ; \hat{e} \text{ is optical polarization}$$

$$H' = -\frac{eA_0}{2m_0} e^{i\vec{k}_{op} \cdot \vec{r} - i\omega t} \hat{e} \cdot \vec{P} \approx -\frac{eA_0}{2m_0} e^{-i\omega t} \hat{e} \cdot \vec{P} ; \text{ (Dipole or long-wavelength approx)}$$

$$\boxed{H'_{ba}} = \langle b | \left(-\frac{eA_0}{2m_0} \hat{e} \cdot \vec{P} \right) | a \rangle = -\frac{eA_0}{2m_0} \hat{e} \cdot \langle b | \vec{P} | a \rangle = \boxed{-\frac{eA_0}{2m_0} \hat{e} \cdot \vec{P}_{ba}}$$

Alternative form:

$$\vec{P} = m_0 \frac{d\vec{r}}{dt} = m_0 \frac{1}{i\hbar} [\vec{r}, H_0] = \frac{m_0}{i\hbar} (\vec{r} H_0 - H_0 \vec{r})$$

$$\langle b | \vec{P} | a \rangle = \frac{m_0}{i\hbar} \langle b | (\vec{r} H_0 - H_0 \vec{r}) | a \rangle = \frac{m_0}{i\hbar} (E_a - E_b) \langle b | \vec{r} | a \rangle = \frac{m_0}{i\hbar} \hbar \omega \langle b | \vec{r} | a \rangle$$

$$\boxed{H'_{ba}} = \left(-\frac{e}{m_0} \vec{A} \right) \cdot (-i\omega m_0) \langle b | \vec{r} | a \rangle = \boxed{-e \vec{E} \cdot \vec{r}_{ba}}$$

EE23z Lecture 6-5

Prof. Ming Wu



Typical Values of Optical Matrix Element

$$H'_{ba} = -\frac{eA_0}{2m_0} \hat{e} \cdot \vec{P}_{cv} = -e \vec{E} \cdot \vec{r}_{ba} = -e i \omega \frac{A_0}{2} \hat{e} \cdot \vec{r}_{ba} \Rightarrow |\vec{r}_{ba}| = \frac{|\vec{P}_{cv}|}{m_0 \omega}$$

The optical matrix element is often expressed in E_p :

$$|\hat{e} \cdot \vec{P}_{cv}|^2 = \frac{m_0}{6} E_p$$

Typical values of E_p (Table K.2 on p.709 of Chuang textbook)

GaAs: $E_p = 25.7$ eV The corresponding $|\vec{r}_{ba}| \sim 0.4$ nm

AlAs: $E_p = 21.1$ eV

InAs: $E_p = 22.2$ eV

InP: $E_p = 20.7$ eV

GaP: $E_p = 22.2$ eV

EE232 Lecture 6-6

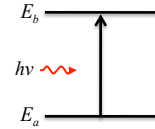
Prof. Ming Wu



Fermi's Golden Rule

Two-level system, monochromatic light:

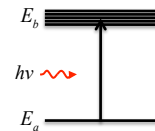
$$W_{ba} = \frac{2\pi}{h} |H'_{ba}|^2 \delta(E_b - E_a - h\omega) \quad \text{Unit: } \left[\frac{1}{\text{J-s}} \text{J}^2 \frac{1}{\text{J}} = \frac{1}{\text{s}} \right]$$



Final state with density of states ρ_b , monochromatic light:

$$W_{ba} = \int \frac{2\pi}{h} |H'_{ba}|^2 \delta(E_b - E_a - h\omega) \rho_b(E_b) dE_b$$

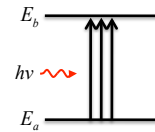
$$\text{Unit: } \left[\frac{1}{\text{J-s}} \text{J}^2 \frac{1}{\text{J}} \frac{1}{\text{cm}^3 \cdot \text{J}} \text{J} = \frac{1}{\text{s-cm}^3} \right]$$



Two-level system, non-monochromatic light:

$$W_{ba} = \int \frac{2\pi}{h} |H'_{ba}|^2 \delta(E_b - E_a - h\omega) \cdot P(h\omega) d(h\omega)$$

$$\text{Unit: } \left[\frac{1}{\text{J-s}} \text{J}^2 \frac{1}{\text{J}} \frac{1}{\text{cm}^3 \cdot \text{J}} \text{J} = \frac{1}{\text{s-cm}^3} \right]$$



where $P(h\omega)$: number of photons per unit volume per energy interval

Prof. Ming Wu



Two-Level System in Non-Monochromatic EM Field

$$W_{ba} = \frac{1}{V} \sum_{k_{op}} \frac{2\pi}{h} |H'_{ba}|^2 \delta(E_b - E_a - h\omega) \cdot 2n_{ph} = \frac{2\pi}{h} |H'_{ba}|^2 n_{ph} N(E_{ba})$$

$$n_{ph} = \frac{1}{e^{h\omega_k/k_B T} - 1} \quad \text{Average number of photons per state (Bose-Einstein distribution)}$$

To calculate density of states for photons: $e^{i\mathbf{k}\cdot\mathbf{r}}$ satisfies periodic boundary condition

$$\omega_k = \frac{kc}{n_r} \quad \text{dispersion relation (\sim E-k relation for electrons)}$$

Number of states with photon energy E_{ba} per unit volume, per energy interval

$$N(E_{ba}) = \frac{2}{V} \int \frac{4\pi k^2 dk}{\left(\frac{2\pi}{L}\right)^3} \cdot \delta(E_b - E_a - h\omega_k) = \frac{8\pi}{(2\pi)^3} \int \left(\frac{n_r \omega_k}{c}\right)^2 \frac{n_r}{c} d\omega_k$$

$$N(E_{ba}) = \frac{8\pi n_r^3 E_{ba}^2}{h^3 c^3}$$

EE232 Lecture 6-8

Prof. Ming Wu



Joint Density of States for Semiconductor

The electron states associated with optical transition are related by conservation of momentum: $k_b \approx k_a \approx k$

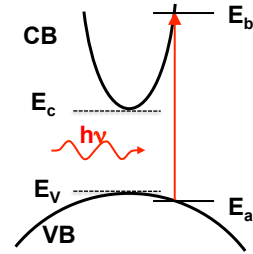
$$E_b = E_C + \frac{\hbar^2 k^2}{2m_e^*} \quad E_a = E_V - \frac{\hbar^2 k^2}{2m_h^*}$$

$$E_b - E_a = (E_C - E_V) + \frac{\hbar^2 k^2}{2} \left(\frac{1}{m_e^*} + \frac{1}{m_h^*} \right)$$

$$\hbar\omega = E_g + \frac{\hbar^2 k^2}{2m_r^*}$$

Joint density of states for the pair of electron states:

$$\rho_r(\hbar\omega - E_g) = \frac{1}{2\pi^2} \left(\frac{2m_r^*}{\hbar^2} \right)^{3/2} \sqrt{\hbar\omega - E_g}$$



EE232 Lecture 6-9

Prof. Ming Wu



Absorption Coefficient

CB completely full, VB completely empty, total upward transition at $\hbar\omega$:

$$R_0(\hbar\omega) = \frac{2}{V} \sum_k \left[\frac{2\pi}{\hbar} |H'_{ba}|^2 \delta(E_b - E_a - \hbar\omega) \right] = \frac{2\pi}{\hbar} |H'_{ba}|^2 \int \frac{2d\vec{k}}{(2\pi)^3} \delta(E_g + \frac{\hbar^2 k^2}{2m_r^*} - \hbar\omega)$$

$$R_0(\hbar\omega) = \frac{2\pi}{\hbar} |H'_{ba}|^2 \rho_r(\hbar\omega - E_g) \quad \text{unit: } \left[\frac{1}{m^3 s} \right]$$

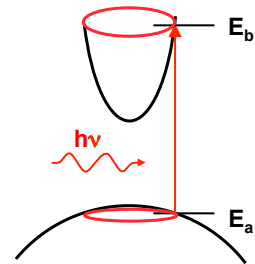
Absorption coefficient

$$\alpha_0(\hbar\omega) = \frac{R(\hbar\omega)}{\text{photon flux}} = \frac{R(\hbar\omega)}{\frac{\epsilon_0 \epsilon_r E_0^2}{2} \frac{1}{n_r \hbar\omega}} = \frac{R(\hbar\omega)}{\frac{\epsilon_0 n_r \omega^2 A_0^2 c}{2 \hbar\omega}} \quad \text{unit: } \left[\frac{1}{m} \right]$$

$$H'_{ba} = -\frac{eA_0}{2m_0} \hat{e} \cdot \vec{P}_{cv}$$

$$\alpha_0(\hbar\omega) = C_0 \left| \hat{e} \cdot \vec{P}_{cv} \right|^2 \rho_r(\hbar\omega - E_g)$$

$$C_0 = \frac{\pi e^2}{n_r c \epsilon_0 m_0^2 \omega} \approx 7 \times 10^9 \quad \text{unit: } \left[\frac{m^2}{kg} \right]$$



EE232 Lecture 6-10

Prof. Ming Wu



Absorption Coefficient for GaAs

$$\alpha_0(\hbar\omega) = C_0 \left| \hat{e} \cdot \vec{P}_{cv} \right|^2 \rho_r(\hbar\omega - E_g)$$

$$n_r := 3.5 \quad \epsilon_0 := 8.854 \cdot 10^{-12} \frac{\text{F}}{\text{m}} \quad \hbar_{\text{bar}} = 1.055 \times 10^{-34} \frac{\text{m}^2 \cdot \text{kg}}{\text{s}} \quad 1 \text{eV} = 1.6 \times 10^{-19} \text{J}$$

$$E_g := 1.42 \text{eV} \quad \omega := \frac{E_g}{\hbar_{\text{bar}}} \quad \omega = 2.154 \times 10^{15} \frac{1}{\text{s}}$$

$$C_0 := \frac{\pi \cdot q^2}{m_0^2 \cdot \omega \cdot \epsilon_0 \cdot c \cdot n_r} \quad C_0 = 4.842 \times 10^9 \frac{\text{m}^2}{\text{kg}}$$

$$E_p := 25.7 \text{eV} \quad \frac{m_0}{6} \cdot E_p = 6.243 \times 10^{-49} \text{kg} \cdot \text{J}$$

$$m_r := \frac{0.067 \cdot 0.5}{0.067 + 0.5} \cdot m_0$$

$$\rho_r(\hbar\omega) := \frac{1}{2\pi^2} \left(\frac{2 \cdot m_r}{\hbar_{\text{bar}}^2} \right)^{\frac{3}{2}} \cdot \sqrt{\hbar\omega - E_g} \quad \rho_r(1.43 \text{eV}) = 6.102 \times 10^{43} \frac{1}{\text{m}^3 \cdot \text{J}}$$

$$\alpha_0(\hbar\omega) := C_0 \cdot \frac{m_0}{6} \cdot E_p \cdot \rho_r(\hbar\omega) \quad \alpha_0(1.43 \text{eV}) = 1.845 \times 10^5 \frac{1}{\text{m}} \quad \alpha_0(1.43 \text{eV}) = 1.845 \times 10^3 \frac{1}{\text{cm}}$$